Omnigenity Optimization

Daniel Dudt, Alan Goodman, Rory Conlin, Dario Panici, Rahul Gaur, Egemen Kolemen

Simons Retreat

Canberra, Australia
December 6, 2023

Outline

1. Motivation
2. Omnigenity Model
3. Tutorial

Why do we care about omnigentiy?

<u>Stellarators are not guaranteed to confine particles!</u>

Toroidal magnetic field gives closed magnetic field lines (hairy ball theorem)

Finite rotational transform helps confine passing particles ☺

But trapped particles can drift radially out and are quickly lost ☹

So we must optimize for confinement of trapped particles (omnigenity)!

But we don’t want to be overly restrictive during the optimization…

Clarification of terminology: OP instead of QI

Isodynamic: Radial drifts of trapped particles vanish everywhere (unrealistic in practice)

Omnigenity: Bounce-averaged radial drifts of trapped particles vanish

Quasi-Symmetry (QS): A special case of omnigenity (sufficient, but not necessary)

Quasi-Isodynamic (QI): Should be synonymous with omnigenity, but is commonly used to refer to the specific case of omnigenity with poloidally closed B contours

Since QI does not have a precise definition and is easily conflated with more general omnigenity, I prefer the terminology omnigenity with poloidal contours (OP)

General omnigenity provides a larger design space for optimization

Previous optimized stellarators have been limited to:

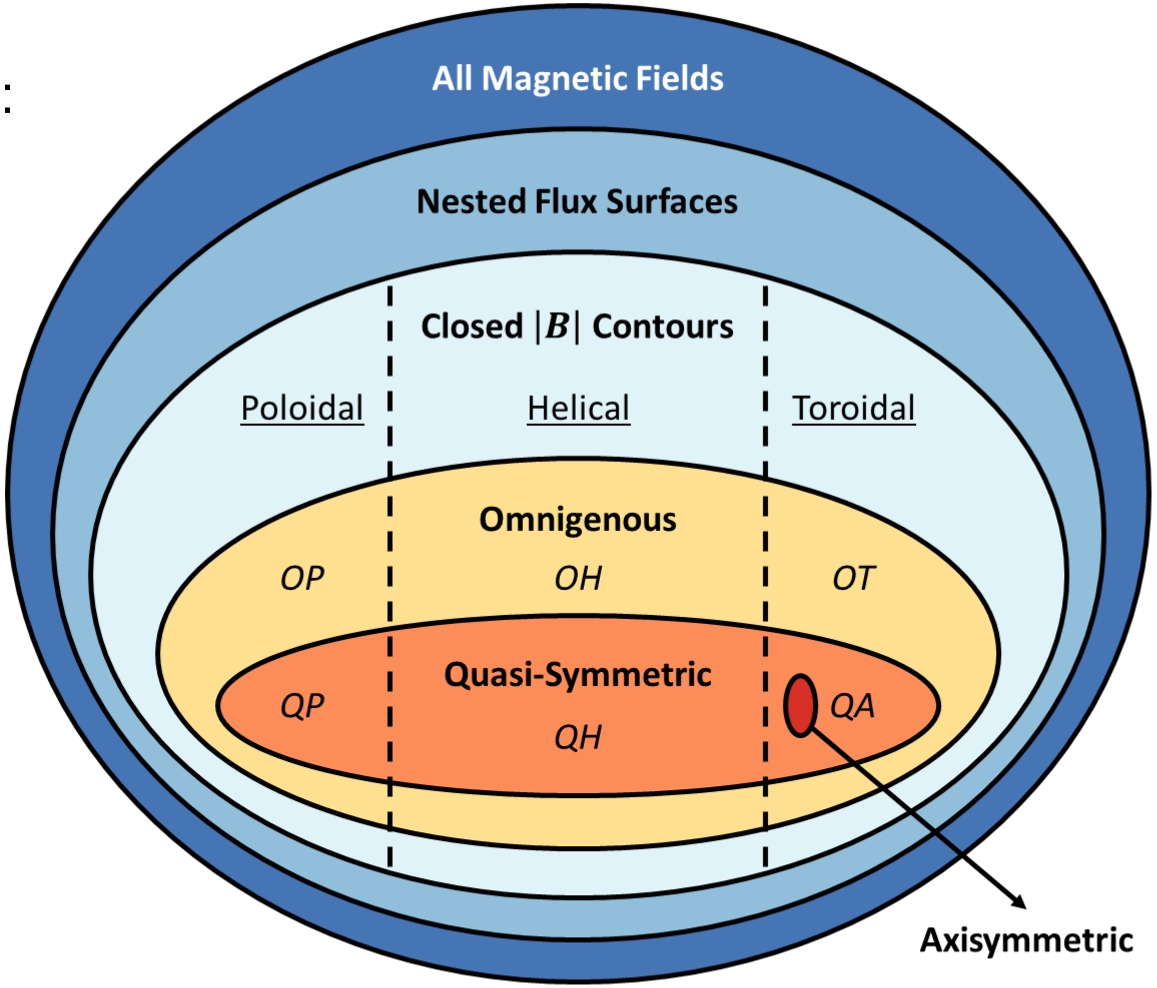
- Axisymmetry (tokamaks)
- Quasi-Symmetry (QA & QH)
- Omnigenity with poloidal contours (OP)

Omnigenity with toroidal and helical contours (OT & OH) are unexplored regions of the design space

Larger design space gives more flexibility for successful multi-objective optimization!

- Neoclassical confinement, turbulence, stability, engineering feasibility, etc.

Dimensions: Omnigenity $\sim \mathcal{O}(N^3)$ vs QS $\sim \mathcal{O}(N^2)$



Plasma Control

Disclaimer: general omnigenity is not exactly possible

Cary & Shasharina [1,2] proved that the only analytic omnigenous fields are quasi-symmetric

Garren & Boozer [3] showed that exact quasi-symmetry is impossible beyond a single surface

This is not a significant restriction in practice, however. Many QA, QH, and OP solutions have been found with very good omnigenity throughout a volume [4-9]

Plunk et al. [10] showed that OT and OH cannot be achieved near the magnetic axis, but this does not preclude them from existing more generally

QP is also impossible near the magnetic axis [11], and is known to be difficult to achieve

Requirements for general omnigenity

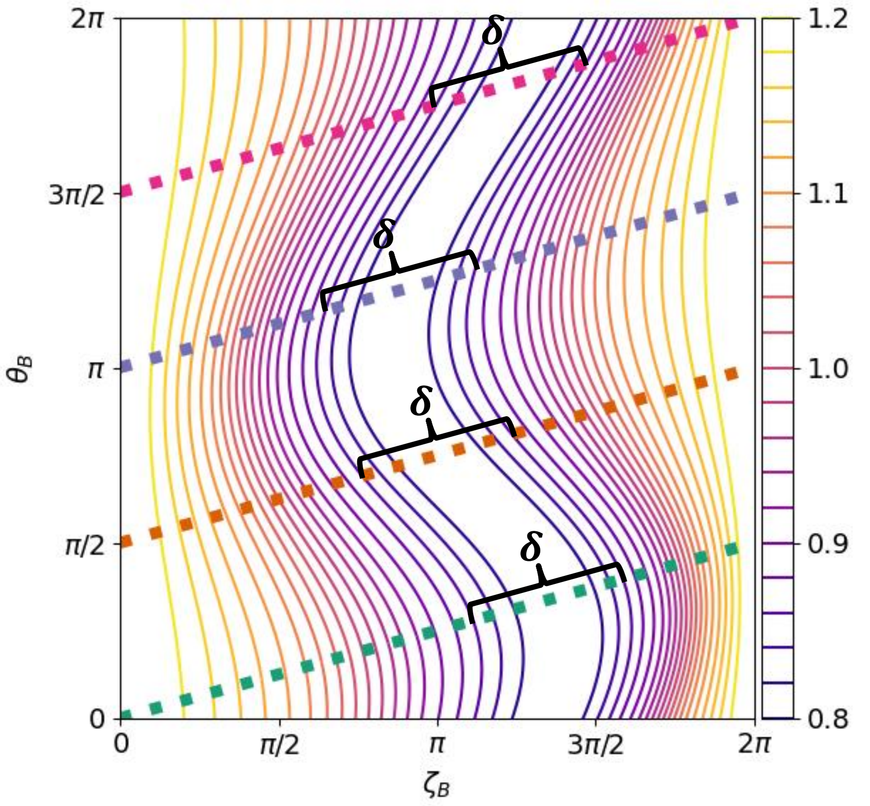
1. Closed contours of magnetic field strength $B = |\boldsymbol{B}|$
2. Straight B_{max} contour in Boozer coordinates*
3. "Bounce distances" δ along a field line between consecutive points with equal B must be independent of the label α

Let the helicity of the B contours be defined by the pair of integers M and N

Field line label: $\alpha = \frac{\theta_B - \iota\zeta_B}{N - \iota M}$

Bounce distance in Boozer coordinates: $\delta = \sqrt{\Delta\theta_B^2 + \Delta\zeta_B^2}$

* And other coordinate systems with a Jacobian that only depends on B and ρ



Overview of the general omnigenity model

1. Construct a perfectly omnigenous magnetic field target $B(\rho, \eta)$
2. Minimize the difference between the MHD equilibrium field and this optimization target:

$$f = B_{\mathrm{eq}}(\rho, \theta_B, \zeta_B) - B(\rho, \eta)$$

The omnigenity target is represented by the parameters $B_{ij} = B\left(\rho_i, \eta_j\right)$

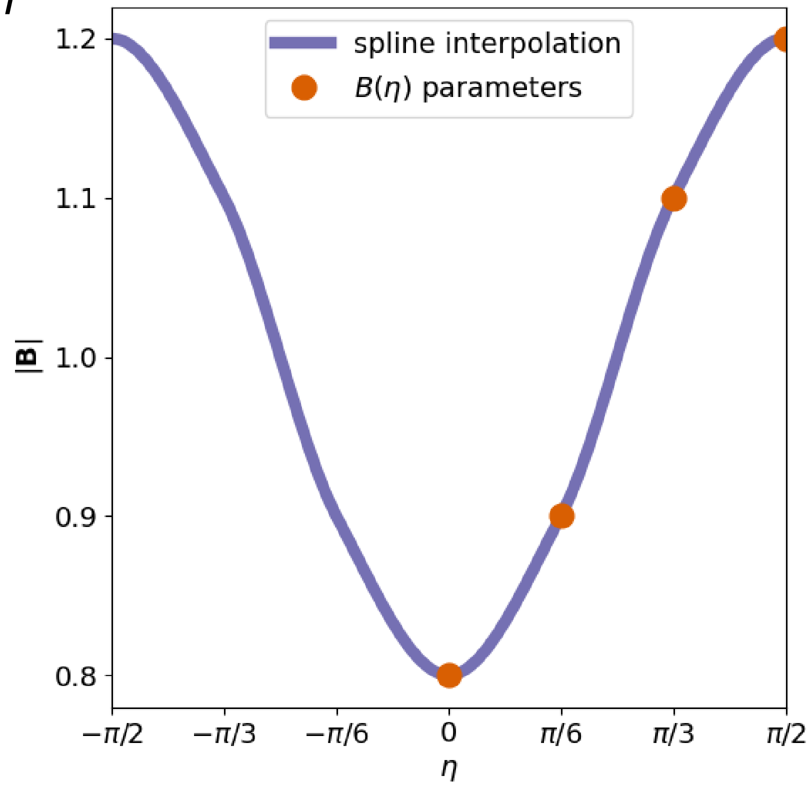
The mapping between Boozer coordinates (θ_B, ζ_B) and the computational coordinates (η, α) is given by another set of parameters x_{lmn}

The model parameters B_{ij} and x_{lmn} can either be fixed or free during the optimization

Magnetic well shape parameterization

The magnetic field strength has the same maximum & minimum along each field line, so it can be written in the form $B = B(\rho, \eta)$

- Contours of constant B are now contours of the new coordinate η
- Similar to Cary & Shasharina approach [1,2]



Requirements on $B(\eta)$:

- $B(0) = B_{\min}$, $B\left(\pm\frac{\pi}{2}\right) = B_{\max}$
- $\left.\frac{\partial B}{\partial \eta}\right|_{\eta=0} = \left.\frac{\partial B}{\partial \eta}\right|_{\eta=\pm\frac{\pi}{2}} = 0$
- $B(\eta)$ monotonically increasing from $B_{\min}$ to $B_{\max}$
- $B(\eta)$ an even function

Same $B(\eta)$ on every field line

Each flux surface can have a different magnetic well shape

$B_{ij} = B(\rho_i, \eta_j)$ parameters are Chebyshev polynomials in ρ and monotonic splines with linearly spaced knots in $\eta \in \left[0, \frac{\pi}{2}\right]$

Plasma Control

Coordinate mapping (Boozer coordinates)

We know the MHD equilibrium magnetic field in Boozer coordinates* $(\rho, \theta_B, \zeta_B)$, but we need it in the computational coordinates (ρ, η, α) to evaluate the residuals $f = B_{\text{eq}}(\rho, \theta_B, \zeta_B) - B(\rho, \eta)$

Define the coordinate mapping as $h(\rho, \theta_B, \zeta_B) = h(\rho, \alpha, \eta)$

In Boozer coordinates it is related to the helicity of the omnigenous field:

$$h(\theta_B, \zeta_B) = \begin{cases} N\zeta_B & \text{for } M = 0 \quad \longleftarrow \text{poloidal contours} \\ -\theta_B + \dfrac{N}{M}\zeta_B & \text{for } M \neq 0 \quad \longleftarrow \text{helical/toroidal contours} \end{cases}$$

Contours of constant h are parallel to the $B_{\max}$ contour (which is straight in Boozer coordinates)

* Really we know B_{eq} in PEST/DESC coordinates (ρ, θ, ζ), but transforming to Boozer coordinates is straightforward

Coordinate mapping (computational coordinates)

In the computational coordinates, we parameterize $h(\rho, \eta, \alpha)$ as

$$h = \underbrace{2\eta + \pi}_{\substack{\text{normalization} \\ \text{to } h \in [0, 2\pi)}} + \sum_{l=0}^{L_\rho} \sum_{m=0}^{M_\eta} \sum_{n=-N_\alpha}^{N_\alpha} x_{lmn} \mathcal{T}_l(2\rho - 1) \mathcal{F}_m(\eta) \mathcal{F}_{nN_{FP}}(\alpha)$$

free parameters

shifted Chebyshev polynomials*

Fourier series

Boundary condition $h\left(\rho, \eta = -\frac{\pi}{2}, \alpha\right)$ to ensure $B_{\max}$ is a straight contour:

$$\sum_{m=0,2,4,\ldots}^{M_\eta} (-1)^{\frac{m}{2}+1} x_{lmn} = 0$$

ρ = flux surface label
η = coordinate along field line
α = field line label

* Zernike polynomials are not required since (ρ,η,α) is not a polar domain

Coordinate mapping (linear system)

Simultaneously solving $\alpha = \frac{\theta_B - \iota\zeta_B}{N - \iota M}$ and $h = \begin{cases} N\zeta_B & \text{for } M = 0 \\ -\theta_B + \frac{N}{M}\zeta_B & \text{for } M \neq 0 \end{cases}$ yields:

$$\begin{bmatrix} \alpha \\ h(\rho,\alpha,\eta) \end{bmatrix} = \begin{bmatrix} \frac{1}{N} & -\frac{\iota}{N} \\ 0 & N \end{bmatrix} \begin{bmatrix} \theta_B \\ \zeta_B \end{bmatrix} \quad \text{for } M = 0; \quad \begin{bmatrix} \alpha \\ h(\rho,\alpha,\eta) \end{bmatrix} = \begin{bmatrix} \frac{1}{N - M\iota} & \frac{-\iota}{N - M\iota} \\ 0 & \frac{N}{M} \end{bmatrix} \begin{bmatrix} \theta_B \\ \zeta_B \end{bmatrix} \quad \text{for } M \neq 0$$

Inverting the system of equations yields:

$$\begin{bmatrix} \theta_B \\ \zeta_B \end{bmatrix} = \begin{bmatrix} N & \frac{\iota}{N} \\ 0 & \frac{1}{N} \end{bmatrix} \begin{bmatrix} \alpha \\ h(\rho,\alpha,\eta) \end{bmatrix} \quad \text{for } M = 0; \quad \begin{bmatrix} \theta_B \\ \zeta_B \end{bmatrix} = \begin{bmatrix} N & \frac{M\iota}{N - M\iota} \\ M & \frac{M}{N - M\iota} \end{bmatrix} \begin{bmatrix} \alpha \\ h(\rho,\alpha,\eta) \end{bmatrix} \quad \text{for } M \neq 0$$

Note: undefined when $\iota = \frac{N}{M}$, but this case is physically uninteresting (field lines parallel to $B_{\max}$)

Coordinate mapping (summary)

We seek to evaluate the residuals $f = B_{\text{eq}}(\rho, \theta_B, \zeta_B) - B(\rho, \eta)$ on a collocation grid $(\rho_i, \eta_i, \alpha_i)$

1. Compute $h_i(\rho_i, \eta_i, \alpha_i)$ from x_{lmn} parameters using $h = 2\eta + \pi + \sum x_{lmn} \mathcal{J}_l(\rho) \mathcal{F}_m(\eta) \mathcal{F}_n(\alpha)$
2. Map the (α_i, h_i) points to Boozer coordinates using $\begin{bmatrix} \theta_B \\ \zeta_B \end{bmatrix} = \begin{bmatrix} N & \frac{M\iota}{N - M\iota} \\ M & \frac{M}{N - M\iota} \end{bmatrix} \begin{bmatrix} \alpha \\ h(\rho, \alpha, \eta) \end{bmatrix}$
3. Evaluate $B_{\text{eq}}(\rho_i, \eta_i, \alpha_i)$ at the corresponding Boozer angles

The collocation grid $(\rho_i, \eta_i, \alpha_i)$ is fixed and linearly spaced, but the corresponding Boozer coordinates vary with the parameters x_{lmn}

Model guarantees constant bounce distances

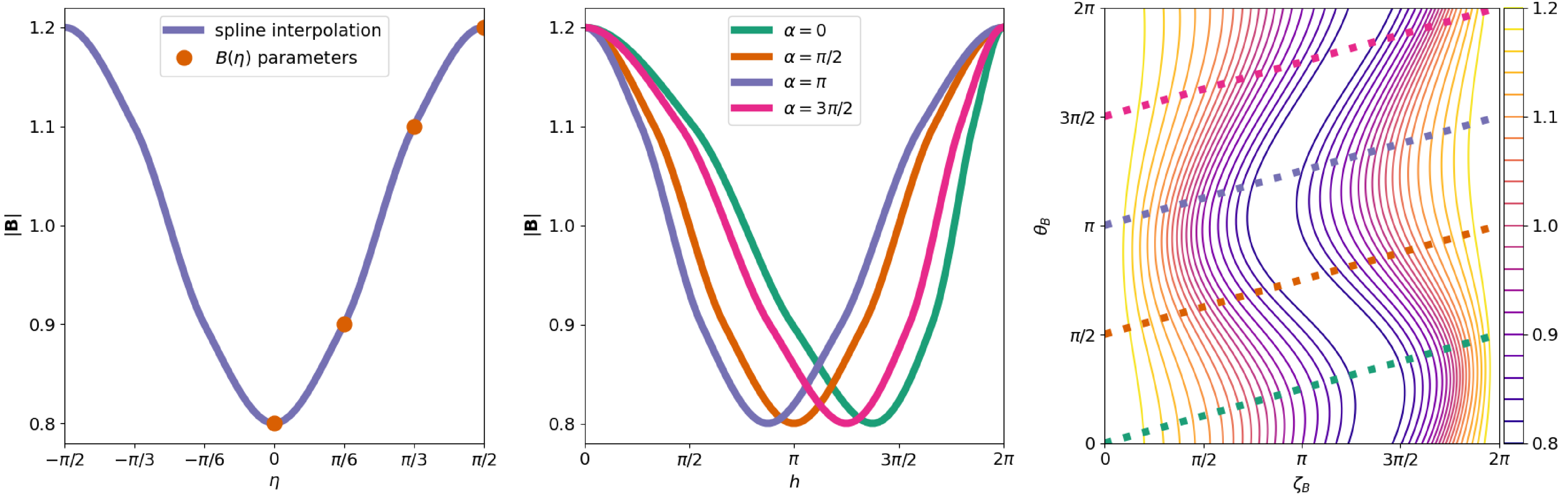
3. "Bounce distances" δ along a field line between consecutive points with equal B must be independent of the field line label α

Bounce points occur at $\pm\eta$ by construction (because $B(\eta)$ is an even function)

$$\begin{aligned}
\delta \propto \Delta h &= h(\rho, +\eta, \alpha) - h(\rho, -\eta, \alpha) \\
&= 4\eta + \sum_{l=0}^{L_\rho} \sum_{m=0}^{M_\eta} \sum_{n=-N_\alpha}^{N_\alpha} x_{lmn} \left[T_l(2\rho - 1) \mathcal{F}_{nN_{FP}}(\alpha) \underbrace{[\mathcal{F}_m(+\eta) - \mathcal{F}_m(-\eta)]}_{\substack{= 0 \text{ because } \mathcal{F}_m(\eta) \text{ are} \\ \text{even functions } \forall\, m \geq 0}} \right] \\
&= 4\eta \\
\therefore \frac{\partial \delta}{\partial \alpha} &= 0
\end{aligned}$$

Bounce distances are solely determined by $B(\rho, \eta)$ target set by B_{ij} parameters, and are always independent of α regardless of x_{lmn} parameters!

Model can parameterize any general omnigenous magnetic field



B_{ij} gives the magnetic well "shape"; x_{lmn} gives the "shift" along each field line

Note: converting to Boozer coordinates requires a value for ι (assuming $\iota = {}^1/_4$ in this plot)

Reduction to quasi-symmetry

The total number of x_{lmn} coefficients is $(L_\rho + 1)(M_\eta + 1)(2N_\alpha + 1)$

Quasi-symmetry corresponds to the condition $N_\alpha = 0$ so that the magnetic well shape is the same on all field lines and $B = B\big(h(\rho, \eta)\big)$

Assuming $L_\rho = M_\eta = N_\alpha$, the dimensions of the parameter space scales as:

- $\sim \mathcal{O}(N^3)$ for general omnigenity
- $\sim \mathcal{O}(N^2)$ for quasi-symmetry

This reveals that QS is only a very small subset of the full omnigenity solution space!

Omnigenity Optimization Tutorial in DESC

https://github.com/PlasmaControl/DESC/blob/dd/omnigenity/docs/notebooks/tutorials/omnigenity.ipynb

Thank you! Questions?

https://doi.org/10.48550/arXiv.2305.08026

https://github.com/PlasmaControl/DESC

References

1. JR Cary, SG Shasharina, Helical plasma confinement devices with good confinement properties. *Phys. Rev. Lett.* **78**, 674–677 (1997).
2. JR Cary, SG Shasharina, Omnigenity and quasihelicity in helical plasma confinement systems. *Phys. Plasmas* **4**, 3323 (1997).
3. DA Garren, AH Boozer, Existence of quasihelically symmetric stellarators. *Phys. Fluids B* **3**, 2822 (1991).
4. M Landreman, E Paul, Magnetic fields with precise quasisymmetry for plasma confinement. *Phys. Rev. Lett.* **128**, 035001 (2022).
5. R Jorge, et al., A single-field-period quasi-isodynamic stellarator. *J. Plasma. Phys.* **88**, 175880504 (2022).
6. K Camacho Mata, GG Plunk, R Jorge, Direct construction of stellarator-symmetric quasi-isodynamic magnetic configurations. *J. Plasma. Phys.* **88**, 905880503 (2022).
7. A Goodman, et al., Constructing precisely quasi-isodynamic magnetic fields. *J. Plasma Phys.* **89**, 905890504 (2023).
8. E Sanchez, JL Velasco, I Calvo, S Mulas, A quasi-isodynamic configuration with good confinement of fast ions at low plasma β (2022).
9. CD Beidler, et al., Demonstration of reduced neoclassical energy transport in wendelstein 7-x. *Nature* **596**, 221–226 (2021).
10. GG Plunk, M Landreman, P Helander, Direct construction of optimized stellarator shapes. part 3. omnigenity near the magnetic axis. *J. Plasma. Phys.* **85**, 905850602 (2019).
11. M Landreman, "Electric fields and transport in optimized stellarators," PhD thesis, MIT (2011).